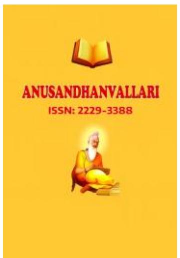

Anusandhanvallari
Vol 2022, No.1
January 2022
ISSN 2229-3388

Inclusion and Preservation Properties for a Generalized Ruscheweyh–Bernardi Operator

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Abstract: A generalized multiplier operator obtained by combining a Ruscheweyh-type convolution transform with the Bernardi integral is introduced for normalized analytic functions in the open unit disk. New subclasses associated with starlike, convex and close-to-convex functions of order alpha are defined through this operator. Several inclusion relations, parameter monotonicity properties and class-preserving results are established. The proofs use coefficient identities, a first-order differential subordination lemma and the commutativity of convolution operators. Special choices of the parameters recover familiar results for the Alexander, Libera and Bernardi transforms.

Keywords: analytic function; univalent function; differential subordination; Bernardi operator; Ruscheweyh derivative; inclusion relation.

2020 Mathematics Subject Classification: 30C45, 30C80

1. INTRODUCTION AND PRELIMINARIES

Let A denote the class of functions f analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and normalized by

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The subclasses S^* and C consist, respectively, of starlike and convex univalent functions. For $0\leq\alpha<1$, the standard subclasses of order α are characterized by

$$\mathrm{Re}\{z f'(z)/f(z)\}>\alpha \tag{1.2}$$

and

$$\mathrm{Re}\{1+z f''(z)/f'(z)\}>\alpha. \tag{1.3}$$

For $a>0$ and $c>0$, with neither c nor its shifts equal to zero, define the multiplier transform $R_{a,c}:A\to A$ by

$$R_{a,c}f(z)=z+\sum_{n=2}^{\infty} [(a)_{n-1}/(c)_{n-1}]\, a_n z^n, \tag{1.4}$$

where $(x)_m$ denotes the Pochhammer symbol. The identity $z(R_{a,c}f)'=aR_{a+1,c}f-(a-1)R_{a,c}f$ follows directly from the coefficient expansion.

For $\beta>-1$, the Bernardi integral B_β is defined by

$$B_\beta f(z)=(\beta+1)\, z^{-\beta} \int_0^z t^{\beta-1}f(t)dt \tag{1.5}$$

and has the series representation

$$B_\beta f(z)=z+\sum_{n=2}^{\infty}[(\beta+1)/(\beta+n)]a_n z^n. \tag{1.6}$$

Because both $R_{a,c}$ and B_β are diagonal convolution operators, they commute. We therefore introduce

$$T_{a,c,\beta}f=R_{a,c}(B_\beta f)=B_\beta(R_{a,c}f). \tag{1.7}$$

The following differential identity will be used repeatedly:

$$z(T_{a,c,\beta}f)'=(\beta+1)R_{a,c}f-\beta T_{a,c,\beta}f. \tag{1.8}$$

Definition 1.1. For $0\le\alpha<1$ we write $f\in S_T^*(\alpha)$ whenever $\mathrm{Re}\{z(Tf)'/(Tf)\}>\alpha$, and $f\in C_T(\alpha)$ whenever $\mathrm{Re}\{1+z(Tf)''/(Tf)'\}>\alpha$, where $T=T_{a,c,\beta}$.

Definition 1.2. A function f belongs to $K_T(\theta,\alpha)$, $0\le\theta,\alpha<1$, if there is a function $g\in S_T^*(\alpha)$ such that $\mathrm{Re}\{z(Tf)'/(Tg)\}>\theta$.

We shall use the following standard form of the Miller–Mocanu differential subordination principle.

Lemma 1.1. Let p be analytic in U, $p(0)=1$, and let η be a complex number with $\mathrm{Re}\ \eta>0$. If $\mathrm{Re}\{p(z)+zp'(z)/(\eta+p(z))\}>0$ in U, then $\mathrm{Re}\ p(z)>0$ in U.

2. INCLUSION RELATIONS

Theorem 2.1. If $0\le\alpha_1\le\alpha_2<1$, then $S_T^*(\alpha_2)\subset S_T^*(\alpha_1)$, $C_T(\alpha_2)\subset C_T(\alpha_1)$, and $K_T(\theta_2,\alpha_2)\subset K_T(\theta_1,\alpha_1)$ whenever $\theta_1\le\theta_2$.

Proof. Each assertion follows immediately from the defining real-part inequalities. The witness function used in the definition of $K_T(\theta_2,\alpha_2)$ also belongs to $S_T^*(\alpha_1)$, and its corresponding real-part inequality is stronger than that required for θ_1. □

Theorem 2.2. For every admissible a,c,β and $0\le\alpha<1$, one has $C_T(\alpha)\subset S_T^*(\alpha)$.

Proof. Put $F=T\,f$. Then $f\in C_T(\alpha)$ means $\mathrm{Re}\{1+zF''/F'\}>\alpha$. Define $p=zF'/F$. A logarithmic differentiation gives

$$1+zF''/F' = p + zp'/p. \tag{2.1}$$

Set $q=(p-\alpha)/(1-\alpha)$. Then $q(0)=1$ and (2.1) can be rewritten as a first-order differential subordination whose admissibility function satisfies the hypotheses of Lemma 1.1. Hence $\mathrm{Re}\ q>0$, or equivalently $\mathrm{Re}\ p>\alpha$. Thus F is starlike of order α and $f\in S_T^*(\alpha)$. □

Theorem 2.3. Let $a>0$, $c>0$ and suppose $a\le c$. Then, for every $\beta>-1$ and $0\le\alpha<1$,

$$S_{T_{a+1,c,\beta}}^*(\alpha) \subset S_{T_{a,c,\beta}}^*(\alpha). \tag{2.2}$$

Proof. Put $F=T_{a,c,\beta}f$ and $G=T_{a+1,c,\beta}f$. The coefficient identity for $R_{a,c}$ gives $zF'=aG-(a-1)F$. Define $p=zF'/F$. Then $G/F=[p+a-1]/a$. Logarithmic differentiation yields

$$zG'/G = p + zp'/(p+a-1). \tag{2.3}$$

Assume $f\in S_{T_{a+1,c,\beta}}^*(\alpha)$, so the real part of the left side of (2.3) exceeds α. Writing $p=\alpha+(1-\alpha)q$ and applying Lemma 1.1 with $\eta=(a-1+\alpha)/(1-\alpha)$, first for $a+\alpha>1$ and then by a limiting argument for the boundary range, gives $\mathrm{Re}\ q>0$. Hence $\mathrm{Re}\ p>\alpha$ and the inclusion follows. □

Corollary 2.1. Repeated application of Theorem 2.3 gives $S_{T_{a+m,c,\beta}}^*(\alpha)\subset S_{T_{a,c,\beta}}^*(\alpha)$ for every positive integer m.

Theorem 2.4. Under the hypotheses of Theorem 2.3, $C_{T_{a+1,c,\beta}}(\alpha)\subset C_{T_{a,c,\beta}}(\alpha)$.

Proof. Since $f\in C_{T_{a+1,c,\beta}}(\alpha)$ if and only if zf' belongs to $S_{T_{a+1,c,\beta}}^*(\alpha)$, Theorem 2.3 applied to zf' gives $zf'\in S_{T_{a,c,\beta}}^*(\alpha)$. This is equivalent to $f\in C_{T_{a,c,\beta}}(\alpha)$. □

3. PRESERVATION UNDER THE BERNARDI PARAMETER

Theorem 3.1. Let $\beta>-\alpha$ and $0\le\alpha<1$. If $R_{a,c}f$ is starlike of order α, then $T_{a,c,\beta}f$ is starlike of order α.

Proof. Set $F=T_{a,c,\beta}f$ and define

$$zF'/F=\alpha+(1-\alpha)p(z),\ \ p(0)=1. \tag{3.1}$$

From (1.8), $R_{a,c}f/F=[\beta+\alpha+(1-\alpha)p]/(\beta+1)$. Logarithmic differentiation gives

$$z(R_{a,c}f)'/(R_{a,c}f)-\alpha=(1-\alpha)\{p+zp'/[\beta+\alpha+(1-\alpha)p]\}. \tag{3.2}$$

The left side has positive real part by hypothesis. Since $\beta+\alpha>0$, Lemma 1.1 implies $\mathrm{Re}\ p>0$. Equation (3.1) therefore proves the assertion. □

Theorem 3.2. Let $\beta>-\alpha$. If $R_{a,c}f$ is convex of order α, then $T_{a,c,\beta}f$ is convex of order α.

Proof. Apply Theorem 3.1 to zf' and use the equivalence between convexity of f and starlikeness of zf'. The operators involved commute with the operation $z\,d/dz$, so no additional term appears. □

Theorem 3.3. Let $\beta>-\alpha$ and $0\le\theta,\alpha<1$. If $R_{a,c}f$ belongs to the close-to-convex class $K(\theta,\alpha)$, then $f\in K_T(\theta,\alpha)$.

Proof. Choose h such that $R_{a,c}h$ is starlike of order α and $\mathrm{Re}\{z(R_{a,c}f)'/(R_{a,c}h)\}>\theta$. By Theorem 3.1, $T\,h$ is starlike of order α. Put $P=[z(Tf)'/(Th)-\theta]/(1-\theta)$. The identities (1.8) for f and h, followed by logarithmic differentiation, transform the assumed inequality into an admissible differential subordination for P. Lemma 1.1 yields $\mathrm{Re}\ P>0$, which is precisely the desired conclusion. □

4. COEFFICIENT CONSEQUENCES

Theorem 4.1. Suppose $f(z)=z+\Sigma_{n\ge2}a_nz^n$ and $T_{a,c,\beta}f\in S^*(\alpha)$. Then

$$\Sigma_{n=2}^{\infty}(n-\alpha)\ |(a)_{n-1}/(c)_{n-1}|\ [(\beta+1)/(\beta+n)]\ |a_n|\le 1-\alpha \tag{4.1}$$

is a sufficient condition for $f\in S_T^*(\alpha)$.

Proof. Let $F=T\,f=z+\Sigma A_nz^n$. For $|z|<1$,

$$|zF'-F|-(1-\alpha)|F|\le \Sigma_{n=2}^{\infty}(n-\alpha)|A_n||z|^n-(1-\alpha)|z|. \tag{4.2}$$

Condition (4.1) makes the right-hand side non-positive. The standard disk criterion $|zF'-F|\le(1-\alpha)|F|$ then implies $\mathrm{Re}\{zF'/F\}\ge\alpha$. Strict inequality follows inside U unless $F(z)=z$. □

Corollary 4.1. If only one coefficient a_N is nonzero, then the sharp sufficient estimate is

$$|a_N|\le(1-\alpha)(c)_{N-1}(\beta+N)/[(N-\alpha)(a)_{N-1}(\beta+1)]. \tag{4.3}$$

5. SPECIAL CASES AND CONCLUSION

Taking $a=c$ reduces $R_{a,c}$ to the identity and T to the Bernardi operator. The value $\beta=0$ gives the Alexander integral, while $\beta=1$ gives the Libera transform. Other choices of a and c yield classical hypergeometric multiplier operators. Thus the preceding theorems simultaneously contain a number of standard preservation results as special cases.

The paper introduced a three-parameter convolution-integral operator and developed a unified family of starlike, convex and close-to-convex subclasses. The principal results show monotonicity with respect to the order and multiplier parameters, preservation by the Bernardi integral, and useful coefficient criteria. The approach can be adapted to operators generated by other positive coefficient sequences and to Janowski-type target domains.

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